Time-Discrete Hedging of Down-And-Out Puts Near the Barrier

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Abstract

Hedging barrier options is challenging. This is especially true when down-and-out puts are considered, where the maximum payoff is reached just before a barrier is hit which would render the claim worthless afterwards. There are several methods proposed by the literature that try to hedge such exotic claims over their whole lifespan. However, all methods may lead to large errors when the underlying is already close to the barrier and the hedge portfolio can only be adjusted in discrete time intervals. Therefore, we focus solely on this most difficult part. This is the first paper that motivates how short-term vanilla call options may be used to dynamically hedge down-and-out puts near the barrier. We consider a dynamic mean-variance hedging method where future option prices are obtained through simulation techniques similar to a stochastic model predictive control approach and compare hedge ratios and errors to a classical delta hedging strategy. Furthermore, we differentiate between hedging discretely in continuous time and hedging in a time period where trading is impossible (overnight gap risk). Consequently, we show in a numerical study where we use both a geometric Brownian motion as well as a jump diffusion process for the underlying that considering overnight gap risk significantly increases both hedge ratios and errors compared to discrete hedging in continuous time near the barrier. We find that short-term vanilla calls can greatly improve hedging performance especially when gap risk is present while the inclusion of jumps only slightly increases hedging errors.

Keywords: exotic option, down-and-out put, time-discrete hedging, mean-variance hedging, Black-Scholes-model, jump-diffusion

1 Introduction

This paper analyzes time-discrete hedging of down-and-out put options near the barrier in discrete time. These barrier options are not only traded over-the-counter, but are also embedded in certain types of retail derivatives, for example bonus certificates (Baule and Tallau, 2011; Hernández et al., 2008), bonus certificates plus (Hernández and Liu, 2014), flex bonus certificates (Hernández et al., 2014) or (multi) barrier reverse convertibles (Wallmeier and Diethelm, 2009). Adequate hedging procedures are therefore highly relevant for banks issuing those kind of structured products.

The payoff of down-and-out put options is discontinuous at the barrier, which make the use of a classical delta hedging problematic when the underlying level approaches the barrier. That is why some approaches suggest hedging exotic options by using a static portfolio of vanilla options of that replicates the exotic payoff at maturity and is zero in case of a barrier hit.[1] However, in theory a continuum of vanilla options may be needed to set up a static hedge for up-and-out calls or down-and-out puts. Engelmann et al. (2006) show with empirical data that certain static hedge strategies for down-and-out puts outperform a delta hedge in a local volatility model while some are worse. Tompkins (2002) shows in a simulation study that neither dynamic hedging nor static hedging leads to satisfactory results for up-and-out calls and that the variability of hedging error is even higher for the static approach than delta hedging. It is also possible to try to combine static and dynamic hedging (İlhan and Sircar, 2006; İlhan et al., 2008; Leung and Lorig, 2015). The biggest challenge, however, is the need to simultaneously unwind the entire hedge portfolio when the barrier is breached. Static hedging of these claims is thus hardly suitable when the spot price is already close to the barrier.

Another strand of literature considers more complex models for delta hedging. An and Suo (2009) compare the performance of delta hedging up-and-out calls on USD/EUR for the Black and Scholes (1973) model, the jump diffusion model by Merton (1976), the stochastic volatility model by Heston (1993) and the stochastic volatility model with jumps by Bakshi et al. (1997). In general, the inclusion of stochastic volatility leads to better hedging performances while adding jumps increases errors. However, it is important to

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[1] See Derman et al. (1995) and Carr and Chou (1997) for the first approaches to static hedging in the BS model and Nalholm and Poulsen (2006) for a unification and extension to general asset dynamics.

note that these hedging error are averages over the whole lifespan of many up-and-out calls. In this paper, we focus on exotic options that are close to the barrier. For short term up-and-out calls in the money, however, An and Suo (2009) find that errors for the Black-Scholes and the Heston models are closer together than for longer maturities or other moneyness levels.[2]

One might argue the problems of a classical delta hedge can be diminished when the hedging frequency is increased near the barrier. However, this is only possible when there is actually a continuous trading of the underlying. On real markets, trading hours are restricted, and the barrier can be breached over night, without the chance to take actions.[3] Furthermore, stock prices may incur discontinuous jumps also during regular trading hours (e.g., Cont, 2001; Cont and Tankov, 2004; Ait-Sahalia and Jacod, 2009; Kou, 2008). Thus, we consider both regular overnight jumps and stochastic jumps in our analysis.

Even for vanilla-options, a classical delta hedge, based on the derivative $\frac{\partial f}{\partial S}$ of the contingent claim value f with respect to the underlying, may lead to considerable hedging errors when hedging is performed in discrete time intervals. Therefore, recent research has focused on methods to calculate a minimum variance delta for vanilla options, which takes a discrete hedging period into account (Vähämaa, 2004; Alexander et al., 2012; Hull and White, 2017). Nian et al. (2018) use a market-data-driven approach, applying kernel functions to determine a delta that minimizes a quadratic empirical loss function in discrete time. However, market data of options is needed for this approach which is not attainable for exotic options.

As outlined before, it is important to consider stochastic jumps near the barrier. With the inclusion of jumps, markets are incomplete and exotic options are not attainable anymore, i.e., they cannot be fully replicated by the underlying. Thus, hedging, in general, cannot be done independently of risk preferences. In these cases quadratic hedging can

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[2]In a similar vein, Gatheral (2006) argues that volatility is almost constant in a small time interval when using stochastic volatility models.

[3]The overnight gap risk has been studied in the case of leverage certificates by Entrop et al. (2009), Entrop et al. (2013) and Baller et al. (2016). In contrast to the down-and-out puts we consider in this paper, leverage certificates feature embedded up-and-out puts with continuous payoffs, which are much easier to hedge.

be used. Risk-minimization and mean-variance approaches are two different methods in this context. While risk-minimization needs a (non self-financing) portfolio that has a terminal value equal to the contingent claim, mean-variance hedging uses self-financing strategies that minimize the mean squared error (MSE) between the contingent claim at maturity and the terminal portfolio value.[4] Both methods can yield the same results as a classical delta hedge with the underlying when it is used to hedge an attainable claim (Pham, 2000). We use a mean-variance hedging strategy that only tries to minimize the MSE for a short period (e.g., 1 day) and not until maturity as this is the most important time period when the underlying is already close to the barrier and a knock out event is likely to occur.

To be able to compute hedging errors, we simulate price changes of both exotic contingent claim and hedge instrument using Monte Carlo simulation techniques. We assume first a geometric brownian motion for the underlying and second a jump diffusion process. This is similar to Bemporad et al. (2010) and Bemporad et al. (2014) where a stochastic model predictive control technique is used resulting in the minimization of the variance of the hedging portfolio at a discrete future point in time. The authors show that their method outperforms delta hedging for up-and-out calls. However, they also use a European call option in addition to the underlying as their hedge instruments. To show that their hedging method is robust to model misspecification, they assume that the real market evolves to the Heston (1993) model with stochastic volatility but use a Black-Scholes log-normal model to generate future prices and show that this outperforms the usage of the "true" Heston model parameters for scenario generation. In subsequent research Bemporad et al. (2011) and Graf Plessen et al. (2019) also include the squared expected hedging error in addition to hedging variance in their optimization.

We add to the literature on hedging barrier options by focusing on down-and-out puts close to the barrier. We show that considering overnight gap risk has a significant impact on mean-variance optimal hedge ratios compared to discrete hedging in continuous time. Furthermore, to our knowledge, this is the first paper that uses opposite vanilla options to dynamically hedge barrier options (i.e., vanilla calls to hedge down-and-out puts).[5]

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[4]See Schweizer (2001) and Pham (2000) for more details of these methods.

[5]Cont et al. (2005) use a vanilla put in addition to the underlying to hedge a barrier put.

We calculate in a numerical analysis that mean-variance hedging leads to much smaller root mean squared errors and value-at-risks than classical delta hedging. Hedging errors can become quite small when continuous trading is possible. However, when gap risk is considered, hedging errors are much larger close to the barrier for all strategies. Again, mean-variance hedging yields better results. However, using short-term vanilla call options as hedge instruments in this case instead of the underlying can further significantly reduce root mean squared errors. If it is possible to find a vanilla call that expires exactly after the trading gap, RMSE and VaR can even be reduced to almost zero. We use both, a geometric Brownian motion as well as jump diffusion process to model future prices. The inclusion of jumps results in slightly higher hedging errors for all strategies but does not change the gain in efficiency when using mean-variance hedging and vanilla calls.

The remainder of the paper is structured as follows: Section 2 describes the hedging problem and our approach for down-and-out puts based on the ideas above and provides a total of six different hedging strategies. These include no hedging, time-continuous Black-Scholes delta hedging as well as mean variance delta hedging using the underlying or vanilla-calls with different maturities. Section 3 shows results for a numerical analysis of hedging errors using the aforementioned strategies in both a Black-Scholes model as well as a Jump diffusion model. In both models we distinguish between continuous trading and overnight gap risk during the hedge period. A short conclusion is given in Section 4.

2   The Hedging Problem

2.1   Mean-Variance Hedging

We consider a down-and-out put (dop) with strike price K, barrier H, and maturity T. The underlying level at time t is denoted S_t. The payoff at maturity is given by

$$\max(K - S_T, 0)\mathbb{1}_{\{S_t \geq H, \, t \in [0,T]\}}, \tag{1}$$

which means that the difference $K - S_T$ is only paid when both the underlying price is below the strike K at maturity and the barrier B has never been hit during the lifetime of the dop.

The main hedging problem is the discontinuity of the payoff at the barrier: The maximum payoff is reached just before the barrier and then drops to zero (see Figure 1, dotted

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black line). As the barrier hit probability increases when the underlying level approaches the barrier from above, the value of the down-and-out put before maturity decreases and reaches zero at the barrier (see Figure 1, black line). Near the barrier, the likeliness of a barrier hit causes the time value to be negative and counteract against the inner value $K - S_t$. Accordingly, the derivative of the value with respect to the underlying level – delta – is positive and very high near but above the barrier. Delta becomes negative for higher underlying levels, similar to a vanilla put option, and is bounded from below by -1. Of course, delta is zero below the barrier. Thus, delta is discontinuous at the barrier, too (see Figure 1, dashed black line).

[Insert Figure 1 about here.]

In theory, hedging options with Black-Scholes delta should lead to a perfect hedge when all corresponding assumptions are met. Hedging in such a way would also minimize the variance of the hedging portfolio.[6] However, trading is only possible at discrete times and thus, price changes are discrete, too. This is especially true when the exchange closes over night or at the weekend. When the barrier is hit at the next trading instance, the down-and-out put is worthless. However, the hedger is still long in the underlying and faces corresponding losses. Consequently, the Black-Scholes delta might lead to large hedging errors. Hull and White (2017) suggest minimizing the variance of discrete-time hedging errors. In their paper they calculate a minimum variance delta empirically for vanilla options. We use the same idea but for down-and-out puts. Thus, our target function to be minimized is the mean squared hedging error:

$$\min_{\delta} \mathbb{E}[\underbrace{(f_{t+\Delta t} - f_t)}_{:=\Delta f_{t+\Delta t}} - \delta \underbrace{(S_{t+\Delta t} - S_t)}_{:=\Delta S_{t+\Delta t}} | \mathcal{F}_t]^2 \tag{2}$$

for a discrete hedging period Δt, where f_t denotes the price of the contingent claim and S_t the price of the underlying at time t.

Equation (2) is similar to the minimization of quadratic hedging errors, i.e., mean variance hedging in incomplete markets.[7] The main difference is that we consider *one* hedging

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[6]See e. g. Bakshi et al. (1997) p. 2034.

[7]First introduced by Föllmer and Sondermann (1986), see Schweizer (2001) or Pham (2000) for an overview of quadratic hedging techniques.

period at time t near the barrier, instead of a self-financing strategy that encompasses the whole lifetime of the option. Hence, all market information until time t (denoted by $\mathcal{F}_t$) is available.

Equation (2) explicitly refers to the underlying itself as the hedging instrument. However, there might be other liquidly traded instruments which are better suited. In particular, short-termed vanilla call options with a strike identical to the barrier of the down-and-out put option could be superior, as illustrated in Figure 1.

For a vanilla call which matures at the end of the hedging period Δt, its payoff below the barrier is zero, inducing a perfect hedge below the barrier. The steep positive slope of the down-and-out put above the barrier can be closely approximated by an appropriate number of vanilla calls. In practice, however, it is not always possible to find a vanilla call option that expires exactly at time Δt. Yet, vanilla calls with short maturities promise to be a superior alternative to the underlying as the hedging instrument. We therefore extend the minimization problem (2) to call options with maturity T_{call}:

$$\min_{\delta} \mathbb{E}[\Delta f_{t+\Delta t} - \delta \Delta C_{t+\Delta t}^{T_{call}}]^2, \tag{3}$$

where $\Delta C_{t+\Delta t}^{T_{call}}$ describes the price difference from t to $t + \Delta t$ of the corresponding vanilla call option.

2.2 Model Setup

In our numerical analysis, we consider a down-and-out put with strike $K = 100$ and barrier $H = 80$. As the delta near the barrier increases when maturity comes closer, we choose a fairly small remaining life time of $T = 20$ days. As the hedging period, we set $\Delta t = 1$ day, as it is common practice to adjust the hedging portfolio on a daily basis. A Monte Carlo simulation is used to generate prices of the underlying in $t + \Delta t$ similar to Bemporad et al. (2014). On one hand, we consider the Black-Scholes model (BS) and assume that the underlying follows a geometric Brownian motion. On the other hand, we assume the jump diffusion model (JD) proposed by Merton (1976) to show that (2) and (3) also work in a more complex model setting. This is because first of all a jump event may have a significant impact on the hedge variance and second of all it is a realistic approximation

of simulating gap risk. Lastly, the model is also incomplete because it is not possible to hedge the jump risk directly.

However, we analyze two different situations regarding the trading of the underlying: First, trading continues throughout the hedging period, and the barrier can be breached at any time between t and $t + \Delta t$. Second, there is no actual trading between t and $t + \Delta t$, and it depends solely on the underlying level at time $t + \Delta t$ whether the barrier is breached or not. The second situation mimics the overnight gap risk between two trading days.[8]

As we are interested in the hedging performances near the barrier, we consider underlying levels S_t between 80 and 82. We investigate a total of six different hedging strategies:

1. Model delta hedge δ (continuous BS / JD)

2. δ^S obtained by (2) using the underlying

3. $\delta^{C_{T_{call}}}$ obtained by (3) using European call options with the following time to maturities:

   (a) 1 day (best case, when available)

   (b) 5 days (normal case, weekly options are available on the German stock index DAX)

   (c) 20 days (worst case, time to maturity identical to DoP)

4. no hedging at all

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[8]Although the usual overnight gap is less than 1 day, exchange holidays and weekends can prolong the time gap between trading opportunities significantly. That is why 1 day seems to be a good proxy for mean gap risk in our numerical analysis.

3 Results

3.1 Black-Scholes Model

3.1.1 Model and Parameters

For simplicity we first assume the stock price to follow a geometric Brownian motion as in Black and Scholes (1973) and Merton (1973). The risk-neutral price dynamic of the underlying is given by

$$\frac{dS_t}{S_t} = rdt + \sigma_{BS}dW_t, \tag{4}$$

where W is a Wiener process, r is the risk free rate and σ_{BS} the volatility.

A major advantage of the Black-Scholes model is the availability of closed-form solutions for both vanilla European options and down-and-out put options. The price of a vanilla European call can be calculated as

$$C^T_{0,BS} = S_0 N(d_1) - Ke^{-rT}N(d_2), \tag{5}$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2_{BS}/2)T}{\sigma_{BS}\sqrt{T}},$$

$$d_2 = d_1 - \sigma_{BS}\sqrt{T}.$$

The price of a down-and-out put option is given by the Reiner and Rubinstein (1991) formula:[9]

$$
\begin{aligned}
pdo_{0,BS} = & - S_0 e^{-rT} N(-x_1) + Ke^{-rT}N(-x_1 + \sigma_{BS}\sqrt{T}) \\
& + S_0 e^{-rT} N(-x_2) + Ke^{-rT}N(-x_2 + \sigma_{BS}\sqrt{T} \\
& - S_0 e^{-rT} (H/S_0)^3 N(y_1) + Ke^{-rT}H/S_0)N(y_1 - \sigma_{BS}\sqrt{T}) \\
& + S_0 e^{-rT} (H/S_0)^3 N(y_2) + Ke^{-rT}(H/S_0)N(y_2 - \sigma_{BS}\sqrt{T}) \\
& + K\left((H/S_0)^{\frac{1}{2}+\lambda}N(z) + (H/S_0)^{\frac{1}{2}-\lambda}N(z - 2\lambda\sigma_{BS}\sqrt{T})\right),
\end{aligned}
\tag{6}
$$

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[9] See Reiner and Rubinstein (1991), p. 32-34 or Haug (2007), p. 152-153.

where

$$x_1 = \frac{\log(S_0/K)}{\sigma_{BS}\sqrt{T}} + \frac{1}{2}\sigma_{BS}\sqrt{T} \qquad y_1 = \frac{\log(H^2/(S_0K))}{\sigma_{BS}\sqrt{T}} + \frac{1}{2}\sigma_{BS}\sqrt{T}$$

$$x_2 = \frac{\log(S_0/H)}{\sigma_{BS}\sqrt{T}} + \frac{1}{2}\sigma_{BS}\sqrt{T} \qquad y_2 = \frac{\log(H/S_0)}{\sigma_{BS}\sqrt{T}} + \frac{1}{2}\sigma_{BS}\sqrt{T}$$

$$\lambda = \sqrt{\frac{1}{4} + \frac{2r}{\sigma_{BS}^2}} \qquad z = \frac{\log(H/S_0)}{\sigma_{BS}\sqrt{T}} + \lambda\sigma_{BS}\sqrt{T}.$$

We fix the risk-free rate at $r = 0.01$ and the volatility at $\sigma_{BS} = 0.2$.

3.1.2 Continuous Trading

We first consider the case (as assumed by the model) when trading continues throughout the 1-day hedging period. For each underlying level S_0, we evaluate $f_0^{BS,cont}(S_0)$ using the Reiner and Rubinstein (1991) formula (6). We then estimate the mean squared hedging error (2) by simulation: We simulate 100,000 realizations of $S_{\Delta t}$ and calculate $f_{\Delta t}^{BS}(S_{\Delta t})$. Given these realizations, it is straightforward to calculate the MSE for arbitrary values of δ and thus find the optimal δ. The simulation approach has the additional advantage that it can be applied to all models we consider. In all simulations we use antithetic variates to reduce simulation variance.

However, we need to consider the possibility that a barrier crossing happened between $t = 0$ and $t = \Delta t$. That is, even if $S_{\Delta t} > H$ it is possible that $S_\epsilon < H$ for some $\epsilon \in [0, \Delta t]$. This probability can be calculated via the Brownian bridge formula as[10]

$$\pi_0 = \exp\left(-2\frac{(S_0 - H)(S_{\Delta t} - H)}{S_0^2\sigma_{BS}^2\Delta t}\right). \tag{7}$$

For each simulated $S_{\Delta t}$, we set $f_{\Delta t}^{BS,cont} = 0$ with probability π_0.

Figure 2 shows the optimal deltas δ, the root mean squared error (RMSE), and the mean hedging error (MHE) for each of the six hedging strategies. As expected, the classical BS-delta δ^{BSM} becomes larger the closer the price of the underlying gets to the barrier at $H = 80$. The difference between the BS-delta and the delta from mean-variance hedging with the underlying or vanilla-call options using (2) becomes very large near the barrier and is almost negligible further away (at $S_0 > 82$). The very small delta near the barrier

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[10]See e.g. Glasserman (2004).

is to be expected as the probability of a barrier crossing is extremely likely and a high delta would lead to a large hedging error in the event of a barrier break.[11]

As expected, RMSE for a Black-Scholes delta hedge are extremely high close to the barrier. That is because the very high position in the underlying leads to a high hedging error, as barrier hit probabilities are almost 1. In this case, even no hedging results in smaller errors, as the value of a dop near the barrier is very small. If S_0 is higher and thus barrier hit probabilities become smaller, the reverse is true: Black-Scholes-hedging converges to the RMSE strategies and no hedging becomes worse and worse. RMSEs for the RMSE minimizing strategies are much smaller close to the barrier and converge at about $S_0 = 81.5$ where errors are nearly identical in magnitude. As deltas are very small the main driver of the hedging error is the rare event when a dop is not knocked out. The reason for this is the steep increase in value of the *pdo* when no barrier crossing happens and $S_{\Delta t}$ is greater than S_0, thus making barrier crossings after $t + \Delta t$ less likely and the derivative more expensive. At higher prices, barrier crossings are less likely an thus trying to minimize RMSE does not yield significant error reduction anymore.

Mean hedging errors are small in comparison. Thus, variance is the dominant driver of RMSE.

[Insert Figure 2 about here.]

Figure 3 provides further insight into the distribution of the hedging error. The upper graph shows the distribution for an initial underlying level 0.5% above the barrier ($S_0 = 80.40$). The further two graphs depict the 99%-VaR for a long and a short hedge, respectively, depending on the initial underlying level S_0. All strategies except no hedging have their modal values close to zero. All RMSE strategies are not as fat tailed as δ^{BS}. Regarding VaR, it is important to differentiate between a long and a short hedge of the down-and-out put, because the source of hedging error that may result in a great loss for the hedger is different in the two cases. The biggest source of hedging error occurs when $S_{t+\Delta t} < H$. However, when a long down-and-out put is hedged, $\Delta f_{t+\Delta t} - \delta \Delta S_{t+\Delta t}$ might be positive because $\Delta S_{t+\Delta t} < 0$, which may yield a large profit for the hedger. If a short down-and-out put is hedged instead, the hedger is long in the underlying or in a call option

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[11]See figure 4 for a visualization of barrier hit probabilities dependent on underlying price.

and loses from this position which might not be compensated by the gain of $-\Delta f_{t+\Delta t}$. The differences between long and short hedge VaR at the 99% level can be seen in the middle and at the bottom of Figure 3. VaR values for δ^{BS} are worse in the continuous world for both long and short hedging but decreases linearly for higher underlying prices. VaR values for the other strategies start at almost zero for a long hedge. This is because the probability of a barrier event is almost 1 in that region which results in an almost worthless dop at time t. Hence, the loss of $\Delta f_{t+\Delta t}$ is also very small. However a potential loss of $\Delta f_{t+\Delta t}$ starts to matter when the underlying price is further away from the barrier at, resulting in a higher price of the dop at time t. The rare event of no barrier crossing and the small deltas close to the barrier lead to high VaR values when a short hedge is considered. That is why in this case VaR values start to decline for the RMSE strategies as the underlying price becomes larger.

[Insert Figures 3 about here.]

3.1.3   Overnight Gap Risk

We now turn to the case when there is no trading during the hedging period. This is the situation when the hedger sets up the hedging portfolio before the exchange closes and has the next chance to react at the following business day. We assume the overnight trading gap to be one calendar day, which is larger than the actual closing period between two regular days but smaller than the weekend closing period.

For valuation purposes, we assume continuous trading after the overnight gap starting at $t = \Delta t$. Thus, we can proceed analogously to the continuous case to estimate the distribution of the down-and-out put at $t = \Delta t$. However, we cannot use the Reiner and Rubinstein (1991) formula to calculate this value at $t = 0$, immediately before the exchange closes, because this formula assumes a continuously monitored barrier, also during the overnight gap between $t = 0$ and $t = \Delta t$. This assumption would lead to an overestimation of knock outs: If the underlying price is extremely close to the barrier, the knock-out probability would tend to one under continuous trading. However, when the underlying *closes* one cent above the barrier, it has a fair chance to jump over night to a level high above the barrier which makes a breach less likely. Figure 4 shows the probability of

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barrier crossings between $t = 0$ and $t = \Delta t$ with respect to S_0. Hence, an overnight gap reduces the knock-out probability and thus increases the value of the down-and-out put.

[Insert Figures 4 about here.]

This is why we calculate the option value $f_0^{BSM,gap} = e^{-r\Delta t}E(f_{\Delta t})$ through Monte-Carlo simulation.

Figure 5 summarizes the results for delta, RMSE, and MHE. The main difference to a continuous world is that now deltas of the mean-variance strategies do not tend to zero near the barrier. This is because now the down-and-out put value in $t = 0$ is significantly higher than in a continuous setting because barrier hit probabilities are much lower (about 50% compared to almost 100%, see Figure 4). The mean-variance delta using the underlying is still smaller than the BS delta. δ^{C^1} for call options that mature at the end of the hedging period of one day is almost identical to δ^{BSM}. This is also plausible as the most problematic scenario when $S_{\Delta t}$ is below H yields now a hedging error of zero as both the call option and the down-and-out put have no payoff. For $S_{\Delta t} > H$ however, the option delta is exactly 1 and thus, yields the same precision as the BS-delta for the "good" case of no barrier break event. δ^{C^5} and $\delta^{C^{20}}$ show nearly the same slope as δ^S but on a higher level. The reason for the slope is that now C^5 and C^{10} still have time values greater than zero in $t = \Delta t$ in contrast to C^1 which again results in a hedging error. The reason for the higher level is that deltas of these options are less than 1. So more options are needed to eliminate the same amount of price change risk than with the underlying as a hedging instrument.

Using δ^S instead of δ^{BSM} cuts RMSE nearly in half close to the barrier, while the advantage of δ^S diminishes further away and becomes nearly zero at prices higher than 82. Using δ^{C^1} is by far the best strategy in terms of RMSE. While it yields the same result far away from the barrier, the error close to the barrier is very low compared to the alternatives. δ^{C^5} and $\delta^{C_{20}}$ still lead to improved performances compared to δ^S, but only for $S_0 < 81.40$. As further calculations show, the reason for this is the different change of delta, i.e., the gamma of the vanilla calls and the down-and-out put. Gamma for vanilla options decreases the further the underlying price is from the strike. However, for the down-and-out put, gamma decreases until 81.40 and then starts to increase. No hedging at all leads to the worst RMSE.

Interestingly, RMSEs starting at about $S_0 = 80.20$ are even smaller than in the world without gap risk until about $S_0 = 81.5$ where errors are nearly identical in magnitude.[12] The reason for this is that on average the value of a dop in $t = 0$ in the gap risk world is higher than in the continuous world as barrier hits are less likely (see figure 4). Thus, deltas are higher resulting in smaller hedging errors when no barrier event occurs.

The mean hedging error is shown at the bottom of Figure 5. Here the order of best to worst is the exact opposite to the order concerning RMSE. This shows the tradeoff between bias and variance. However, all mean hedging errors are rather small considering the scale. So most of the RMSE is again induced by the variance.

[Insert Figure 5 about here.]

Figure 6 shows the distribution of the hedging error as well as VaR values for short and long hedging. In general, distributions seem to be more skewed as in the continuous world. There is a clear shift of modal values depending on hedging strategy. While no hedging results in a negative mode, the mode of δ^{C_1} is nearly zero, whereas δ^{C_5} and $\delta^{C_{20}}$ yield positive modes. Also the distributions of no hedging and $delta^{BSM}$ have fat tails and are skewed to the right.

Clearly, δ^{C_1} performs best in both cases of long and short hedge value at risk. As expected, short hedging with δ^{BSM} leads to much bigger VaR-values than long hedging. Also the differences between hedging strategies are more severe in the former case. δ^S reduces already from -4.8 to -2. However, δ^{C_5} can again half the former result. δ^{C_1} leads to a VaR between 0.05 (at $S_0 = 80.01$) and 0.17 (at $S_0 = 82$).

[Insert Figure 6 about here.]

3.2 Jump Diffusion Model

3.2.1 Model and Parameters

One problem of the Black-Scholes-Model is that it assumes the log return to be normally distributed. However, a normal distribution cannot accurately describe the tails of empirically observed distributions of stock prices (fat tails) (Fama, 1965; Cont, 2001). This

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[12]See section 3.1.2.

might lead to a severe underestimation of hedging errors especially near the barrier. One way to address this issue is to allow stock prices to jump. Merton (1976) incorporates jumps in a mixture of both continuous and jump processes. The resulting process is as follows:

$$\frac{dS_t}{S_t} = (r - \lambda\kappa)dt + \sigma_M dW_t + dJ_t, \tag{8}$$

where r is the risk free rate, σ_M the volatility conditional on no jumps, W is a Wiener process and J is an independent compound Poisson process that both determines and counts the arrivals of jumps and determines the jump amplitude by a given distribution of jump heights, which is independent of the occurrence of jumps. The probability that n jumps occur in a small time interval dt is λdt and $1 - \lambda dt$ that no jumps occur, where λ is the mean number of jumps per unit time. The probability that more than one jump occurs in dt is therefore approximately zero. That is why (8) can be rewritten as

$$\frac{dS_t}{S_t} = \begin{cases} (\alpha - \lambda k)dt + \sigma_M dW_t & \text{if a jump occurs} \\ (\alpha - \lambda k)dt + \sigma_M dW_t + (Y_t - 1) & \text{if no jump occurs} \end{cases}, \tag{9}$$

where Y_t is the percentage change of the underlying caused by a jump and $\kappa := E(Y_t - 1)$ the expectation of $Y_t - 1$. We assume Y_t to be normally distributed with parameters μ_J and σ_J.

The solution of (8) is given by

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma_{BS}^2)t + \sigma_{BS}W_t} \prod_{j=1}^{N_t} Y_j, \tag{10}$$

with N_t counting the number of jumps until t.[13]

A closed-form solution is only available for vanilla European options, while exotic options, like the down-and-out put, need to be evaluated numerically. A European call option can be calculated as a weighted sum of BS calls as follows:

$$C_{0,JD}^T = \sum_{i=0}^{\infty} \frac{\exp(-(1+\kappa)\lambda((1+\kappa)\lambda)^i}{i!} C_{0,BS_i}^T \tag{11}$$

with

$$C_{0,BS_i}^T = C_{0,BS}^T(r := r - \lambda\kappa + i\log(1+\kappa)/T, \sigma_{BS} := \sigma_M + i + \sigma_J^2/T) \tag{12}$$

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[13]For further details see Merton (1976) or Tankov and Voltchkova (2009).

Each BS call C_{0,BS_i}^T is conditioned on the arrival of exactly i jumps. The weights are given by the probability that a Poisson random variable with parameter $\lambda(1 + \kappa)$ takes value i. For a better comparability, we want the variance of returns in the Black-Scholes setting to equal the variance in the Jump Diffusion setting. The variance in the Jump diffusion model is given by (Matsuda, 2004):

$$Variance_{JD} = \sigma_M^2 + \lambda\sigma_J^2 + \lambda\mu_J^2 \equiv Variance_{BS}. \tag{13}$$

We therefore set

$$\sigma_{JD}^2 = \sigma_{BS}^2 - \lambda\mu_J^2 - \lambda\sigma_J^2. \tag{14}$$

By choosing $\sigma_{BS} = 0.2$, $\lambda = 5$, $\mu_J = \log(0.97)$, $\sigma_J = 0.02$ we obtain $\sigma_M \approx 0.1827$.

3.2.2 Continuous Trading

We use Eq. (10) to simulate $S_{\Delta t}^i$ for $i = 1, \ldots, N = 100{,}000$. To calculate f_0^{JD} and $f_{\Delta t}^{i,JD}$, we use a Monte-Carlo simulation with 100,000 replications and 20 equidistant steps till maturity per iteration. It is important to note that barrier crossings can happen between each two time steps of each path. That is why we calculate π_j according to (7) for each time step. The probability that the barrier is still intact after the whole path is therefore $1 - \prod_{j=1}^{20} \pi_j$. Multiplying this probability at the end of each path with the corresponding payoff leads to significant reduction in discretization error.

However, we also need to take into account that barrier crossings can happen between $t = 0$ and $t = \Delta t$ when simulating $S_{\Delta t}$. As of now, $f_{\Delta t}^{i,JD}$ is just the price of a dop when the underlying in $t = \Delta t$ equals $S_{\Delta t}^i$ without considering the path from S_0 to $S_{\Delta t}^i$. That is why we consider a total of 8 time steps in this time interval. Then we calculate $\prod_k^8 \pi_k$ and draw a Bernoulli random number B^i that is 0 with that probability and 1 otherwise. We then set $f_{\Delta t}^{i,JD,cont} = B^i \cdot f_{\Delta t}^{i,JD}$ for each $i = 1, \ldots, 100000$.

Figures 7 and 8 summarize the results for discrete trading in continuous time in the Jump Diffusion model. They are very similar to the results of discrete trading in the BSM-model, but have higher deltas. MHE's are more negative and VaR values are much worse when using δ^{JD} to hedge.

[Insert figure 7 and 8 here.]

3.2.3 Overnight Gap Risk

Calculation of $f_{\Delta t}^{JD}$ is the same as in section 3.2.2. However, since no trading is possible between $t = 0$ and $t = \Delta t$, i. e. no barrier crossings are possible, $f_0^{JD,gap}$ is simply the discount value of $E(f_{\Delta t}^{JD})$.

Results for modeled gap risk in Merton's jump diffusion model are shown in figures 9 and 10. The slope and order of deltas are almost identical to the corresponding BS counterpart. The same is true for RMSE. Interestingly, hedging with δ^{JD} can now lead to higher RMSE than no hedging very close to the barrier. The reason for this is that δ^{JD} takes only volatility of the diffusion process into account and neglects the added volatility due to jumps. Accordingly, volatility is too low leading to higher dop prices in the model since vega is negative close to the barrier. Consequently, deltas are up to 0.5 higher as in the Black-Scholes model leading to larger errors when the barrier is hit. Another difference is that δ^{JD} now leads to lower MHE than all three call based hedging variants. Distributions of hedge errors and VaR values are similar to gap risk in BSM, again, with one exception: δ^{JD} performs worse in the Jump Diffusion setting than no hedging close to the barrier.

[Insert figure 9 and 10 here.]

4 Conclusion

We implemented a mean-variance method for time-discrete hedging of down-and-out puts near the barrier. Additionally to using the underlying we motivated the use of short-term vanilla-call options as hedge instruments. We showed in a numerical analysis that classical Black-Scholes delta hedging yields the worst results. In a continuous setting, hedging performances of the other strategies are considerably better but quite similar amongst each other and can even yield very small RMSEs. However, all proposed hedging methods lead to higher errors when no trading during the hedge period is possible, i.e., when overnight gap risk is present. In these cases, discrete mean-variance hedging using vanilla calls significantly outperforms hedging with the underlying. The performance gain over the usage of the underlying is dependent on the call's maturity (the closer to maturity, the better). Results do not change significantly when jumps are included in the simulation

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except for value-at-risk. The proposed method can easily be extended to be used with other models (e.g., stochastic volatility, double exponential models). As up-and-out calls share the same complexity and discontinuity at the barrier, we expect that short-term vanilla put options can reduce hedging errors in the same manner as vanilla calls for down-and-out puts. Whether the proposed strategies work empirically with real data is yet to be determined. For future research and in order to obtain a feasible long term strategy for the whole lifespan of a down-and-out put, one could try to implement a static-dynamic hedging strategy similar to İlhan and Sircar (2006) but substituting the underlying with a short-term vanilla call option as a dynamic instrument when the underlying is close to the barrier.

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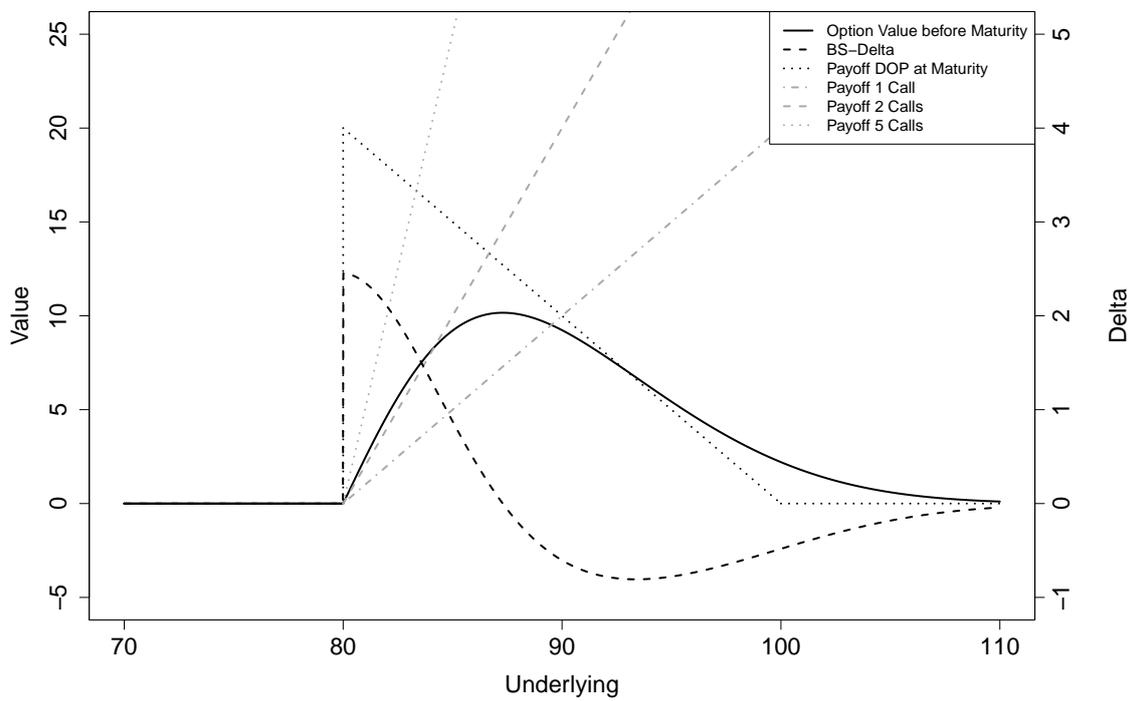
Figure 1. *Value, corresponding Black-Scholes-delta and payoff of a down-and-out put 20 days before maturity with strike $K = 100$ and barrier $H = 80$. Additionally, payoffs of 1, 2 and 5 vanilla European calls with strike equal to barrier of down-and-out put are shown.*
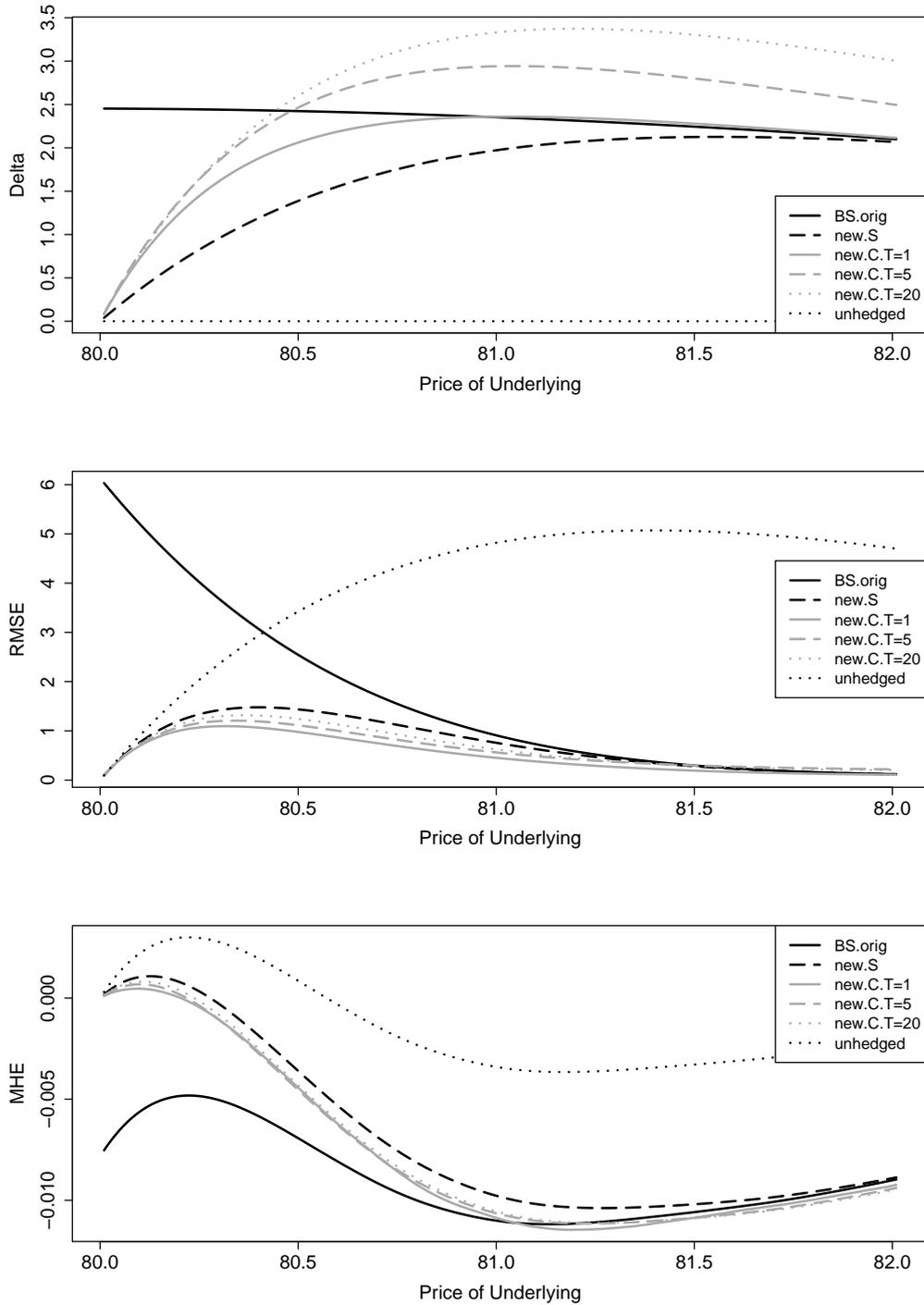
Figure 2. *Deltas, root mean squared errors and mean hedging errors of hedging a down-and-out put 20 days before maturity with strike $K = 100$ and barrier $H = 80$ for 1 trading day depending on price of the underlying at initiation of the hedge portfolio in a Black-Scholes model with discrete hedging in continuous time for the following different hedging strategies: Model delta (BS.orig), MSE delta using the underlying (new.S), MSE delta using vanilla calls with time to maturity of 1 (new.C.T=1), 5 (new.C.T=5) and 20 (new.C.T=20) days as well as no hedging (unhedged).*

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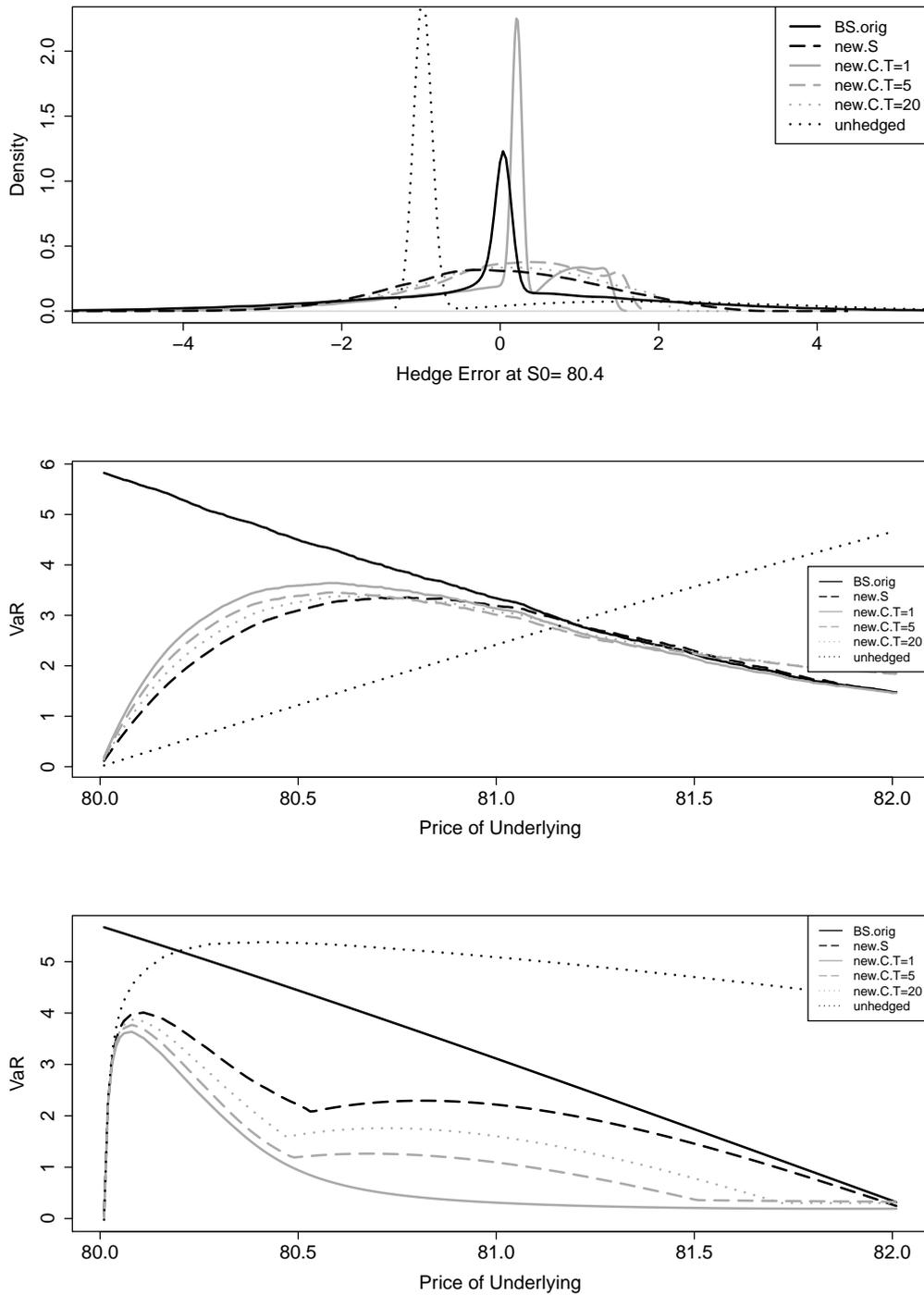
Figure 3. *Density of hedge error 0.5 % above the barrier and 99%-Value-At-Risk for long and short hedge of hedging a down-and-out put 20 days before maturity with strike $K = 100$ and barrier $H = 80$ for 1 trading day depending on price of the underlying at initiation of the hedge portfolio in a Black-Scholes model with discrete hedging in continuous time for the following different hedging strategies: Model delta (BS.orig), MSE delta using the underlying (new.S), MSE delta using vanilla calls with time to maturity of 1 (new.C.T=1), 5 (new.C.T=5) and 20 (new.C.T=20) days as well as no hedging (unhedged).*

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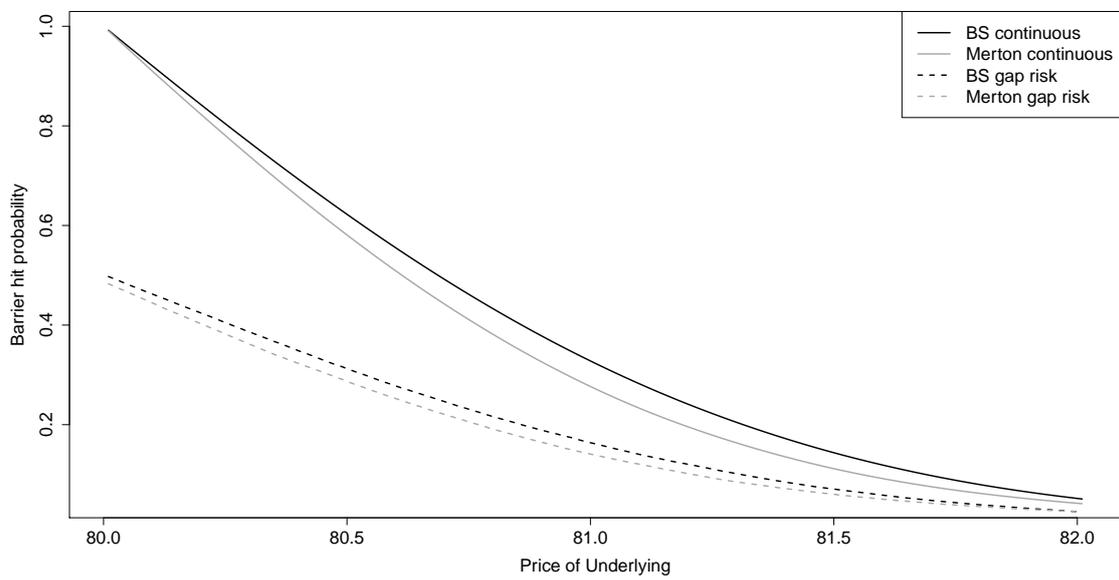
Figure 4. *Mean probability of barrier crossing between $t = 0$ and $t = 1$ day resulting in zero payoff of down-and-out put 20 days before maturity with strike $K = 100$ and barrier $H = 80$ in a continuous (BS continuous) and gap risk (BS gap risk) Black-Scholes model as well as a continuous (Merton continuous) and gap risk (Merton gap risk) jump-diffusion model .*
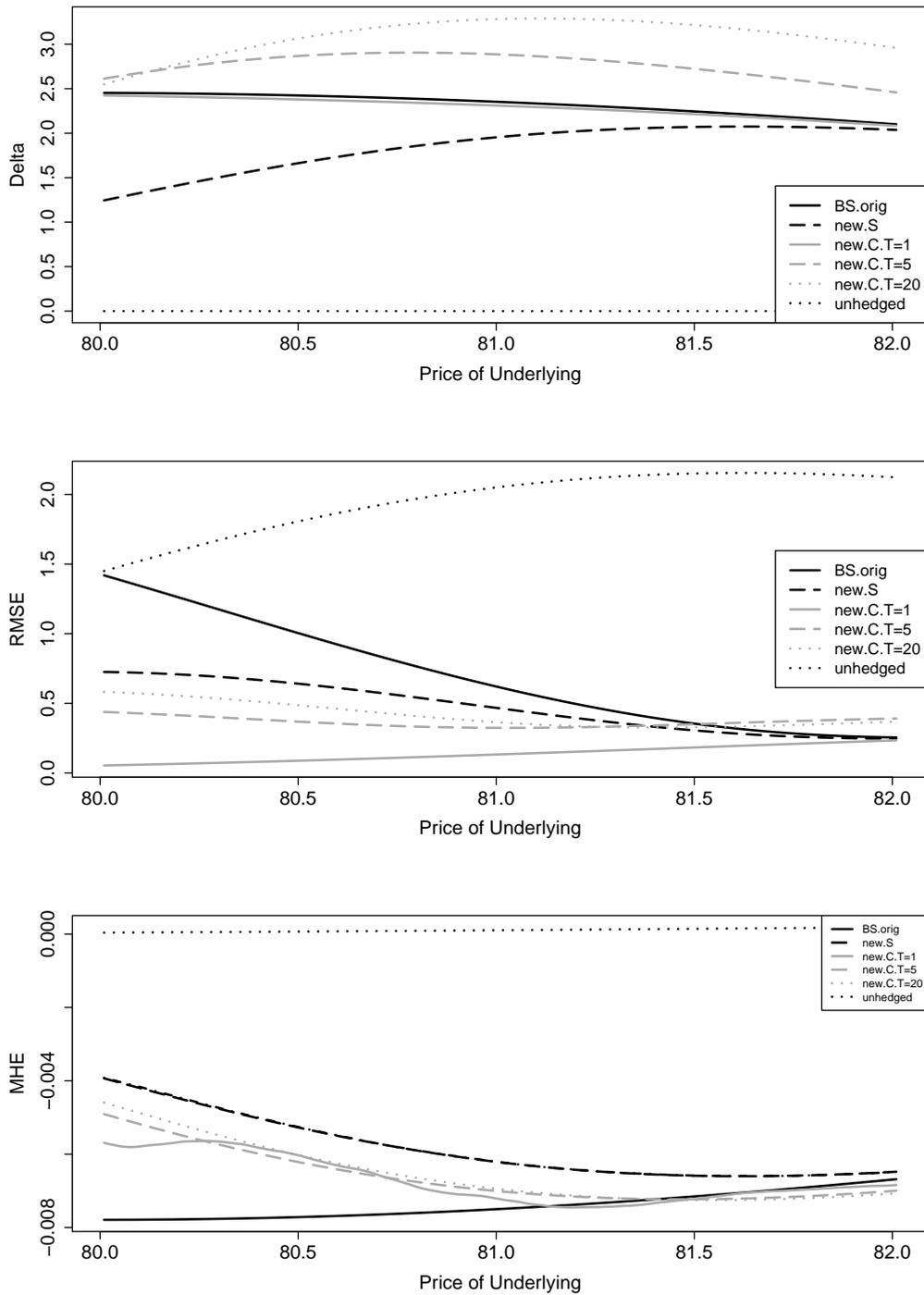
Figure 5. *Deltas, root mean squared errors and mean hedging errors of hedging a down-and-out put 20 days before maturity with strike $K = 100$ and barrier $H = 80$ for 1 trading day depending on price of the underlying at initiation of the hedge portfolio in a Black-Scholes model with gap risk for the following different hedging strategies: Model delta (BS.orig), MSE delta using the underlying (new.S), MSE delta using vanilla calls with time to maturity of 1 (new.C.T=1), 5 (new.C.T=5) and 20 (new.C.T=20) days as well as no hedging (unhedged).*
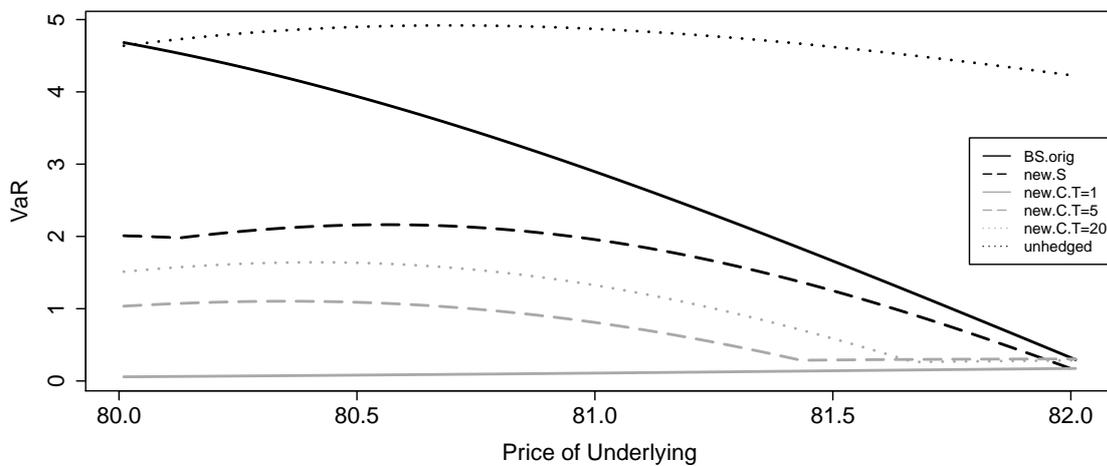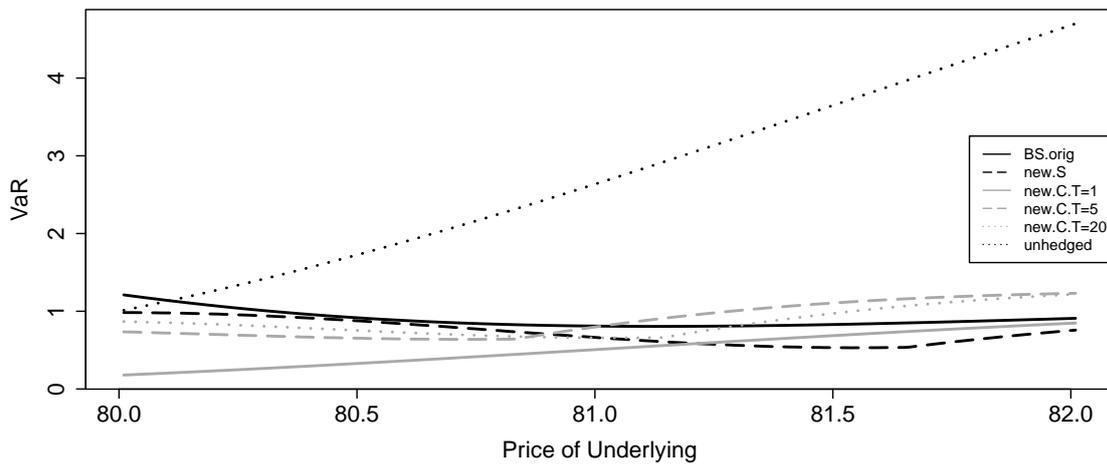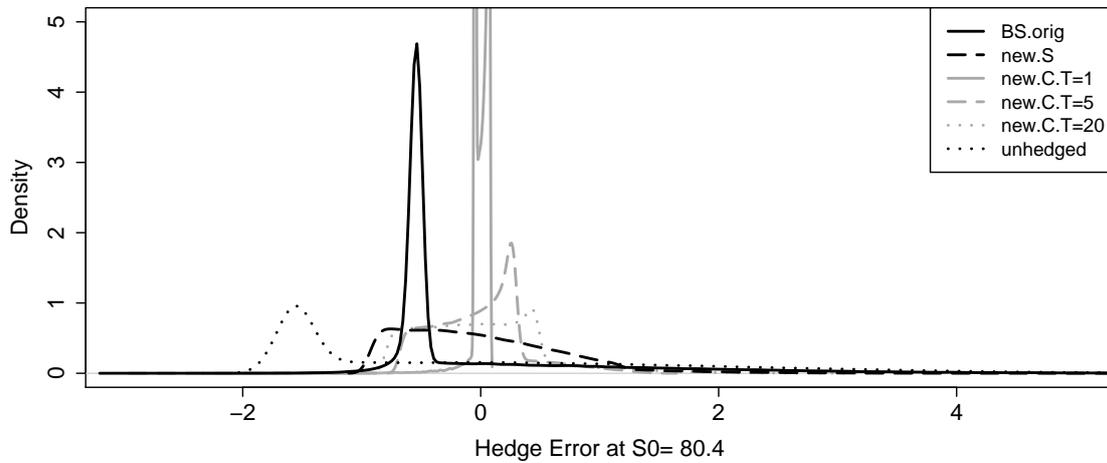
Figure 6. *Density of hedge error* 0.5 *% above the barrier and* 99%-*Value-At-Risk for long and short hedge of hedging a down-and-out put 20 days before maturity with strike K = 100 and barrier H = 80 for 1 trading day depending on price of the underlying at initiation of the hedge portfolio in a Black-Scholes model with gap risk for the following different hedging strategies: Model delta (BS.orig), MSE delta using the underlying (new.S), MSE delta using vanilla calls with time to maturity of 1 (new.C.T=1), 5 (new.C.T=5) and 20 (new.C.T=20) days as well as no hedging (unhedged).*
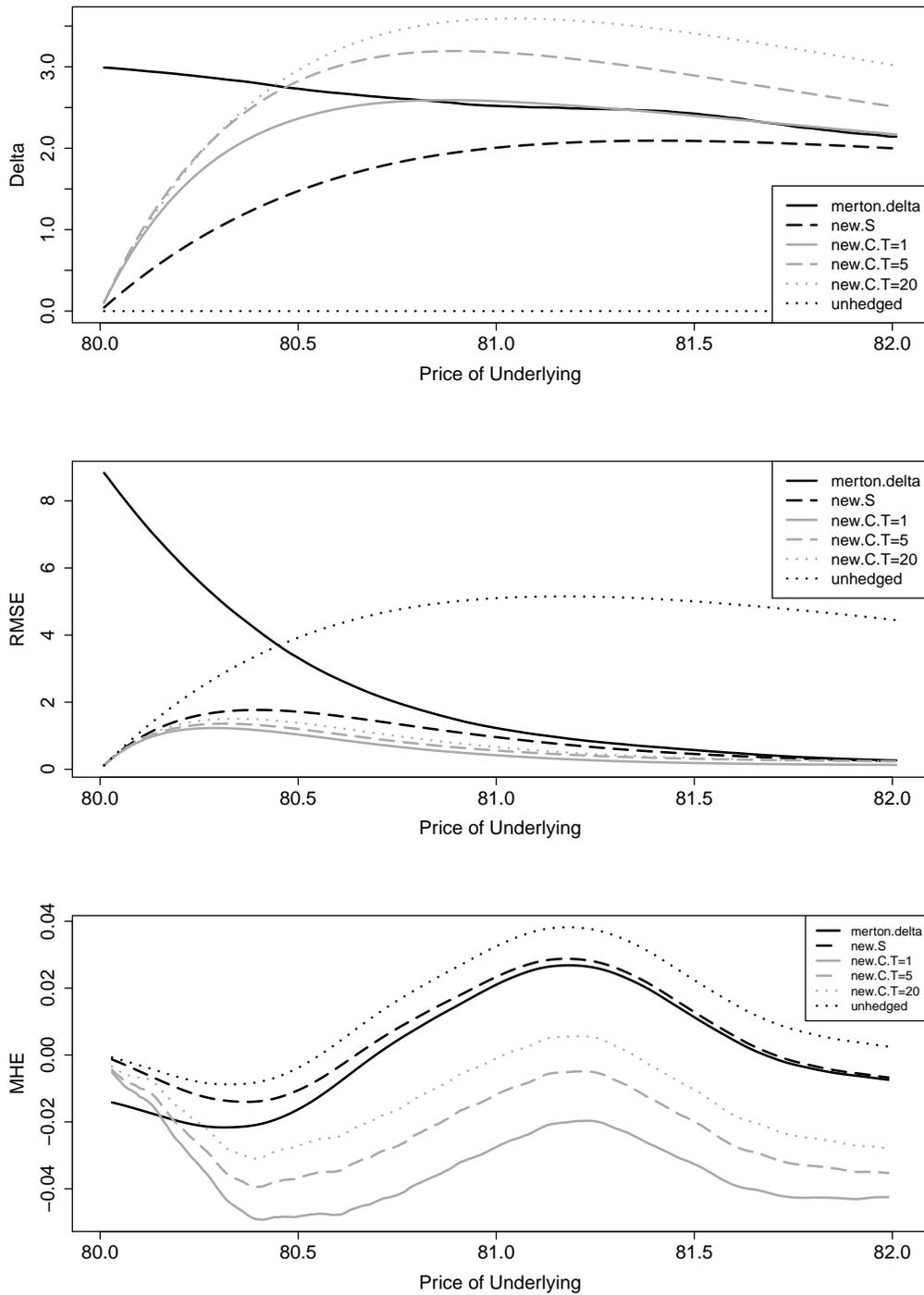
Figure 7. *Deltas, root mean squared errors and mean hedging errors of hedging a down-and-out put 20 days before maturity with strike $K = 100$ and barrier $H = 80$ for 1 trading day depending on price of the underlying at initiation of the hedge portfolio in a jump-diffusion with discrete hedging in continuous time for the following different hedging strategies: Model delta (BS.orig), MSE delta using the underlying (new.S), MSE delta using vanilla calls with time to maturity of 1 (new.C.T=1), 5 (new.C.T=5) and 20 (new.C.T=20) days as well as no hedging (unhedged).*
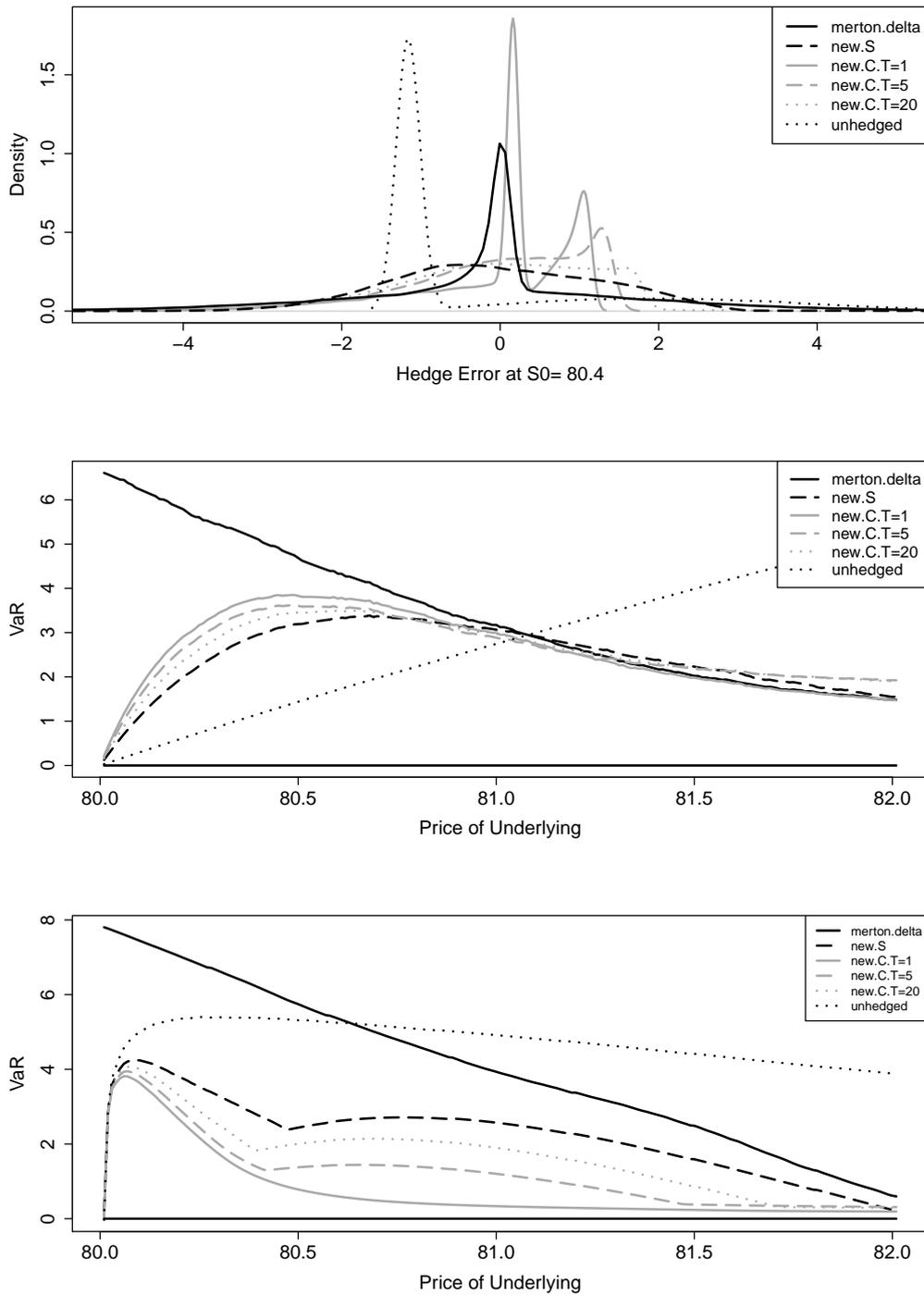
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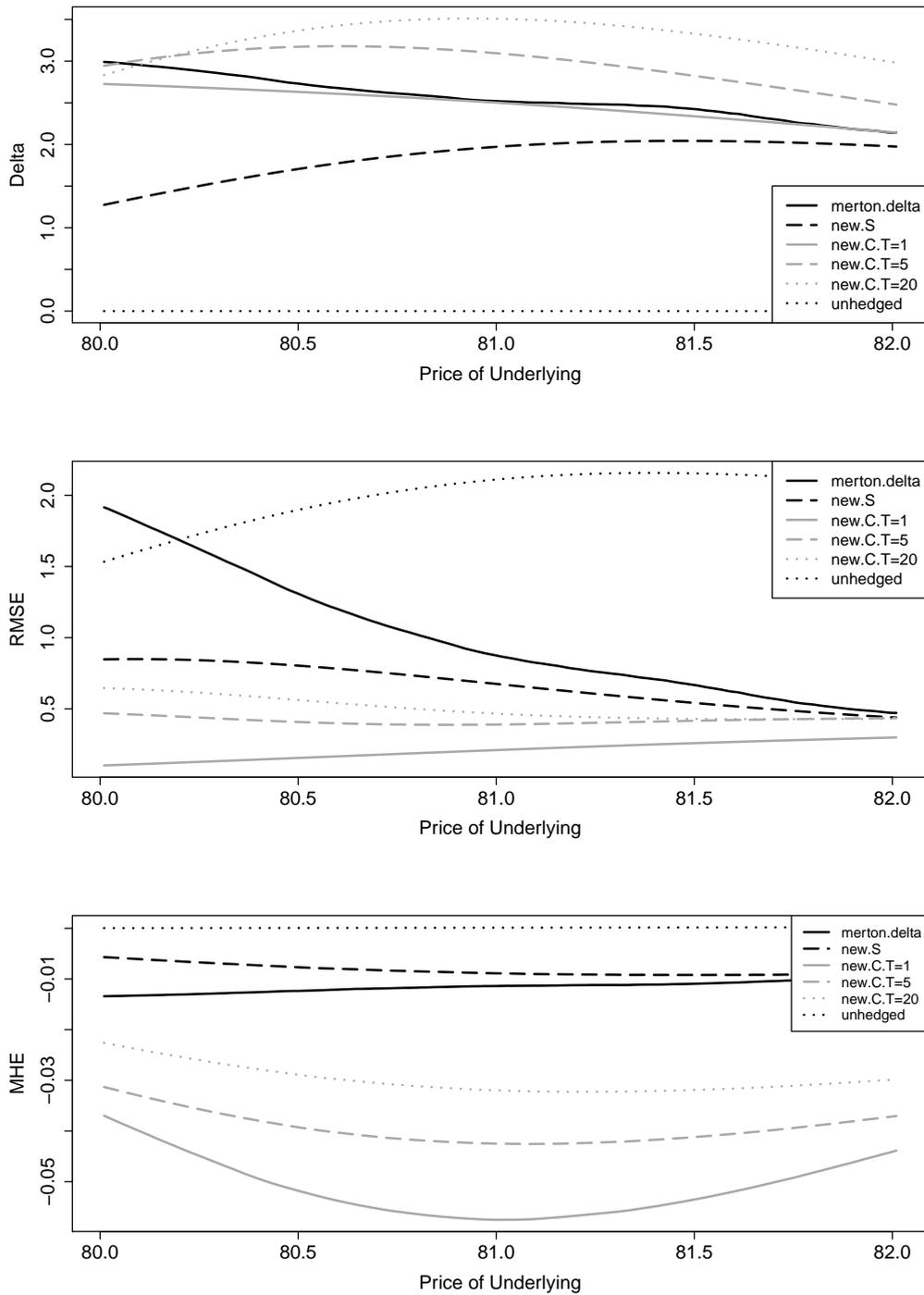
Figure 9. *Deltas, root mean squared errors and mean hedging errors of hedging a down-and-out put 20 days before maturity with strike $K = 100$ and barrier $H = 80$ for 1 trading day depending on price of the underlying at initiation of the hedge portfolio in a jump-diffusion with gap risk for the following different hedging strategies: Model delta (BS.orig), MSE delta using the underlying (new.S), MSE delta using vanilla calls with time to maturity of 1 (new.C.T=1), 5 (new.C.T=5) and 20 (new.C.T=20) days as well as no hedging (unhedged).*
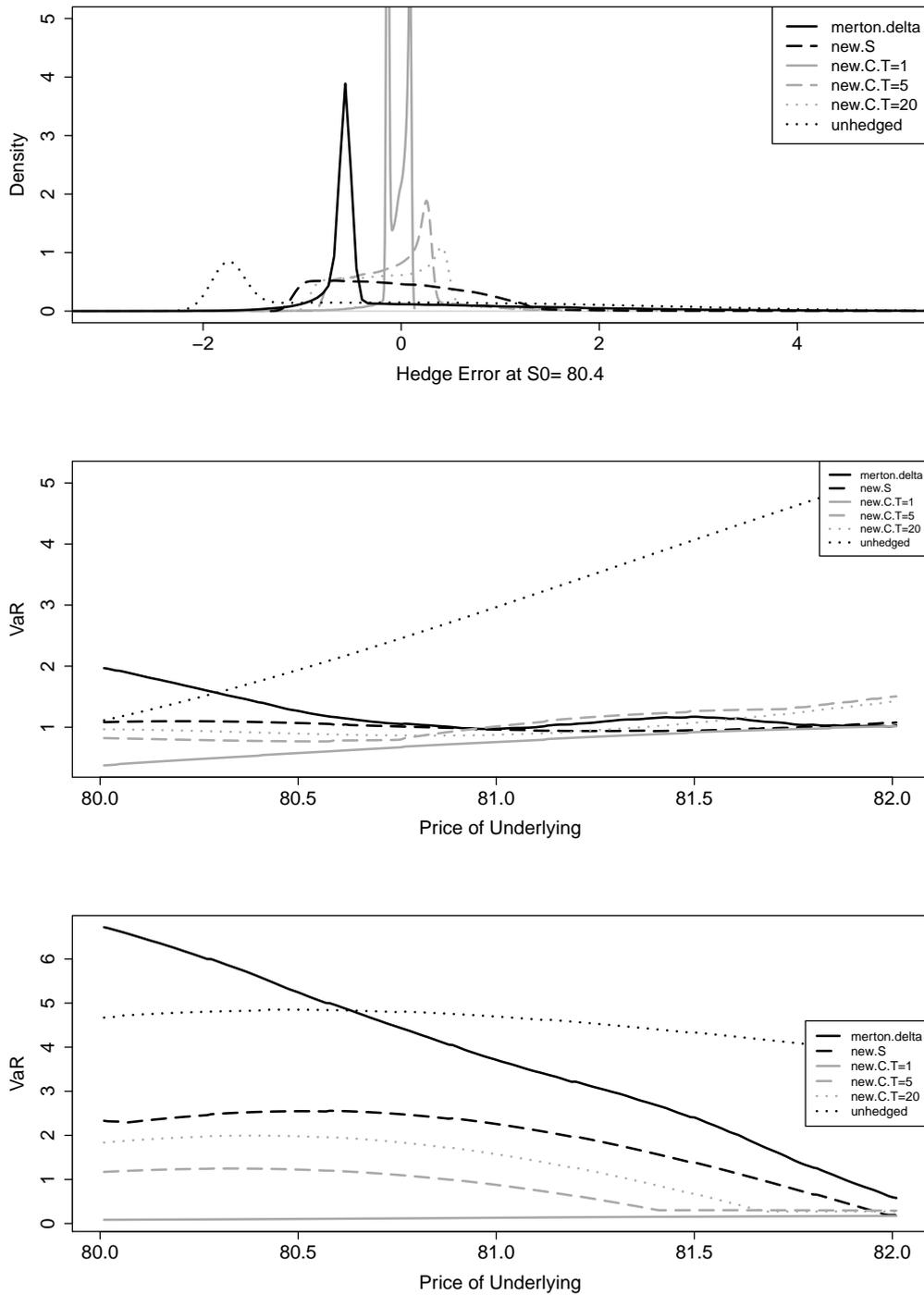
Figure 10. *Density of hedge error* 0.5 *% above the barrier and* 99%*-Value-At-Risk for long and short hedge of hedging a down-and-out put 20 days before maturity with strike* $K = 100$ *and barrier* $H = 80$ *for 1 trading day depending on price of the underlying at initiation of the hedge portfolio in a Merton model with gap risk for the following different hedging strategies: Model delta (BS.orig), MSE delta using the underlying (new.S), MSE delta using vanilla calls with time to maturity of 1 (new.C.T=1), 5 (new.C.T=5) and 20 (new.C.T=20) days as well as no hedging (unhedged).*